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PAPER - 3 Part

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Logarithmic Test

Th Suppose that $U_n > 0$ and that

$$Lt\left(n \log \frac{U_n}{U_{n+1}}\right) = K$$

Then the series is convergent if $k > 1$ and divergent if $k < 1$.

Proof Let us ~~suppose~~ compare the given series $\sum U_n$ with the auxiliary series

$$\sum V_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

whose nth term $V_n = \frac{1}{n^p}$. We know that the series $\sum V_n$ is convergent if $p > 1$ and is divergent if $p \leq 1$.

Now $$\frac{V_n}{V_{n+1}} = \frac{1}{n^p} \bigg/ \frac{1}{(n+1)^p}$$

$$= \frac{(n+1)^p}{n^p} = \left(\frac{n+1}{n}\right)^p = \left(1+\frac{1}{n}\right)^p$$

Case I Suppose that $\sum V_n$ is convergent and hence $p > 1$. Then by the comparison test $\sum U_n$ is convergent

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if $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$

i.e. if $\dfrac{u_n}{u_{n+1}} > \left(1+\dfrac{1}{n}\right)^p$

i.e. if $\log \dfrac{u_n}{u_{n+1}} > p \log\left(1+\dfrac{1}{n}\right)$

i.e if $\log \dfrac{u_n}{u_{n+1}} > p\left(\dfrac{1}{n} - \dfrac{1}{2n^2} + \cdots\right)$

i.e if $n \log \dfrac{u_n}{u_{n+1}} > p - \dfrac{p}{2n} + \cdots$

i.e if $Lt\left(n \log \dfrac{u_n}{u_{n+1}}\right) > p (>1)$

i.e if $Lt\left(n \log \dfrac{u_n}{u_{n+1}}\right) > 1$

Case II Suppose that $\sum u_n$ is divergent and hence $p < 1$. Then $\sum u_n$ is divergent if $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$ i.e if $Lt\left(n \log \dfrac{u_n}{u_{n+1}}\right) < p < 1$ as shown before. Hence the theorem.